1901001101020001
EXAMINATION NOVEMBER 2024
MASTER OF ARTS (MATHEMATICS) (PART - I) (EXTERNAL)
COMPLEX ANALYSIS - LEVEL 2

[**Time: As Per Schedule]** **[Max. Marks: 100]**

Instructions:
1. Fill up strictly the following details on your answer book
  a. Name of the Examination : **MASTER OF ARTS (MATHEMATICS) (PART - I) (EXTERNAL)**
  b. Name of the Subject : **COMPLEX ANALYSIS - LEVEL 2**
  c. Subject Code No : **1901001101020001**
2. Sketch neat and labelled diagram wherever necessary.
3. Figures to the right indicate full marks of the question.
4. All questions are compulsory.
5. There are five questions in this questions paper.

Seat No:

Student's Signature

Q.1 A. If the complex numbers $\sin x + i\, cos\, 2x$ and $\cos x - i \sin 2x$ are complex conjugate to each other, then find the value of x. **7**

B. Show that **7**
  1. $|z_1 + z_2| \leq |z_1| + |z_2|$
  2. $||z_1| - |z_2|| \leq |z_1 - z_2|$
  3. $|1 - \overline{z}_1 z_2|^2 - |z_1 - z_2|^2 = 1 + |\overline{z}_1|^2\, |z_2|^2 - (|\overline{z}_1|^2 + |z_2|^2)$

C. Prove that the modulus of the product of two complex numbers is the product of their moduli **6**

OR

A. Let z_1, z_2, be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and $|z_2|$ has negative real part than prove that $\frac{z_1+z_2}{z_1-z_2}$ has purely imaginary expression. **7**

B. Express the following complex numbers in the form x + iy, where x, yare real numbers. **7**
(a) $(-1 + 3i)^{-1}$ (b) (1+i)i(2-i) (c) (7 + πi)(π + i)

C. Describe geometrically the sets of point's z satisfying the following conditions. **6**

1. |z-i+3|=5
2. Im z ≥ 1
3. |z+2i|≤1

Q.2 A. Define analytic function. Find the analytic function $f(z)$ of which the real part is $u = e^x (x\cos y - y\sin y)$. Also, find v such that $f(z) = u + iv$. **7**

B. If w= f(z). Assume that f is differentiable at z, and g is differentiable at w then prove that (g o f) is differentiable at z, and (g o f)' (z) =g' (f(z)) f'(z). **7**

C. State and Prove Cauchy- Riemann theorem. **6**

OR

A. Prove that $u = \sin x \cos hy + 2\cos x \sin hy + x^2 - y^2 + 4xy$ is harmonic function. **7**

B. If f is analytic in domain D then prove the following **7**

1. If f' (z)=0 in D f is constant.
2. If one of |f|, Re(f), Im(f) is constant in D, f is constant.

C. Prove that the function $f(z) = xy + iy$ is everywhere continuous but not analytic. **6**

Q.3 A. Prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|u|^n = n\,(n-1)|u|^{n-2}|f'(z)|^n$ **7**

B. Evaluate: $I = \int_\gamma f(z)dz\ ;\ f(Z) = z^2$ Where γ is along the horizontal segment 0 to 1 and then by vertical segment from 1 to 1+2i. **7**

C. Evaluate: **6**

1. $I = \int_\gamma y\, dz,$ whereyis the circle |z|=R.
2. $I = \int_\gamma \bar{z}dz,$ Where γ circumference of the square [0, 1] x [0, 1].

OR

A. State and prove Cauchy - Goursat Theorem **7**

B. Let φ be a complex valued function which is continuous in an open set D containing an arc γ. Then ∀z ∉ γ the function F_n is defined by $F(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)} d\zeta; n = 1,2,3,...$ is analytic and satisfies the equation $F'n(z) = n\, F_{n+1}(z)$ or equivalently $F^k(z) = k! \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{k+1}} d\zeta;$ $k = 1,2,3, ...$ with $F_1(z) = F(z)$ **7**

C. State and prove Morera's Theorem **6**

Q.4 A. Define Laurent's series about $z_{0.}$ Also Distinguish removable singularity, pole and essential singularity using Laurent's series. **7**

B. Prove that the zeros of an analytic function is isolated. **7**

C. Let $f(z)$ be an analytical function in a simply connected region D. And $a_1,\ a_2, ... a_n, ...$be a sequence of zeros having a as its limit point, a being the interior point of D. Then prove that either $f(z)$ vanishes identically or else has an isolated essential singularity at $z = a$. **6**

OR

A. If $f\ (z)$ and $g(z)$ are analytic in Ω and if $f(z) = g(z)$ on a set which has a limit point in Ω, then prove that $f(z)$ is identically equal to $g(z)$. **7**

B. State and prove Cauchy's residue theorem. **7**

C. Prove that $\lim_{z \to \infty} -zf(z) = Res\ (z = \infty)$ provided $f\ (z)$ is analytic at $z = \infty$, **6**

Q.5 1. Find the residue of the following functions **7**

1. $f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}\, at\ z = 1,2,3$
2. $f(z) = \frac{1}{(z^2+1)^3}\, at\ z = -i$

2. Find residue of $\phi(z)$ = cotz at the points $z_n = n\pi$ for $n = 1,2, \ldots$. what is the nature of singularity at $z = \infty$? Justify your answer. **7**

3. Discuss the nature of singularities of the following functions **6**

1. $f(z) = \frac{1}{z(1-z^2)}$
2. $f(z) = \frac{z}{1+z^4}$
3. $f(z) = \frac{sin\ z}{(z-\pi)^2}$

OR

A. If $f(z)$ has pole of order m at $z = a$, then prove that the residue at a is the $\lim_{z\to q} \frac{1}{(m-1)!} \frac{a^{n-1}}{dz^{m-1}} [(z-a)^m f(z)]$. **7**

B. If function is analytic at every point and finite at infinity, then it must be constant. **7**

C. If AB is the arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z-a| = r$ and if $\lim_{z\to a} \int_{AB} (z-a)f(z) = k$ then prove that $\lim_{z\to a} \int_{AB} f(z) = i(\theta_2 - \theta_1)k$. **6**

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